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STOCHASTIC DERIVATIVES FOR FRACTIONAL DIFFUSIONS

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In this paper, we introduce some fundamental notions related to the so-called *stochastic derivatives* with respect to a given σ-field $\mathcal{Q}$. In our framework, we recall well-known results about Markov Wiener diffusions. We afterwards mainly focus on the case where X is a fractional diffusion and where $\mathcal{Q}$ is the past, the future or the present of X. We treat some crucial examples and our main result is the existence of stochastic derivatives with respect to the present of X when X solves a stochastic differential equation driven by a fractional Brownian motion with Hurst index $H > 1/2$. We give explicit formulas.

1. Introduction. There exist various ways to generalize the notion of differentiation on deterministic functions. We may think about fractional derivative or differentiation in the sense of the theory of distributions. In both cases, we lose a dynamical or a geometric interpretation as tangent vectors, velocities for instance. In this present work, we want to construct derivatives on stochastic processes which conserve a dynamical meaning. Our goal may be motivated by the stochastic embedding of dynamical systems introduced in [3]. This procedure aims at comprehending the following question: how to write an equation which contains the dynamical meaning of an initial ordinary differential equation and which extends this dynamical meaning on stochastic processes? We refer to [4] for more details.

Unfortunately, for most of the stochastic processes used in physical models, the limit

$$\frac{Z_{t+h} - Z_t}{h}$$

does not exist pathwise. What can we do to give a meaning to this limit? One of the main available tool is the "quantity of information" we can use to calculate it, namely a given σ-field $\mathcal{Q}$. The idea is that one can remove the divergences which appear by doing some means in the computation. This fact can be achieved by studying the behavior when h goes to zero of the

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conditional expectation:

$$\mathrm{E}\left[\frac{Z_{t+h}-Z_t}{h}|\mathcal{Q}\right].$$

Such objects were introduced by Nelson in his dynamical theory of Brownian diffusion [10]. For a fixed time t, he calculates a forward (resp. backward) derivative with respect to a given σ-field $\mathcal{P}_t$ which can be seen as the past of the process up to time t (resp. $\mathcal{F}_t$, the future of the process after time t). The main class with which he can work turns out to be that of Wiener diffusions.

The purpose of this paper is, on one hand, to introduce notions to study the above mentioned quantities for general processes and, on the other hand, to treat some examples. We mainly study these notions on solutions of stochastic differential equations driven by a fractional Brownian motion with Hurst index $H \geqslant \frac{1}{2}$. In particular, we recall results on Wiener diffusions (case $H = \frac{1}{2}$) in our framework. We prove that for a suitable σ-algebra, the stochastic derivatives of a solution of the fractional stochastic differential equation exist and we are able to give explicit formulas.

Our paper is organized as follows. In section 2, we recall some now classical facts on stochastic analysis for fractional Brownian motion. In section 3, we introduce the fundamental notions related to the so-called *stochastic derivatives*. In section 4, we study stochastic derivatives of Nelson's type for fractional diffusions. We show in section 5 that stochastic derivatives with respect to the present turn out to be adequate tools for fractional Brownian motion with $H > \frac{1}{2}$. We treat also the more difficult case of a fractional diffusion.

2. Basic notions for fractional Brownian motion. We briefly recall some basic facts about stochastic calculus with respect to a fractional Brownian motion. One refers to [13] for further details. Let $B = (B_t)_{t\in[0,T]}$ be a fractional Brownian motion with Hurst parameter $H \in (0,1)$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We mean that B is a centered Gaussian process with the covariance function $\mathrm{E}(B_sB_t) = R_H(s,t)$, where

$$R_H(s,t) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right). \tag{1}$$

If $H = 1/2$, then B is a Brownian motion. From (1), one can easily see that $\mathrm{E}|B_t - B_s|^2 = |t-s|^{2H}$, so B has $\alpha-$Hölder continuous paths for any $\alpha \in (0,H)$.

2.1. *Space of deterministic integrands.* We denote by $\mathcal{E}$ the set of step $\mathbb{R}-$valued functions on [0,T]. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathcal{H}} = R_H(t,s).$$

We denote by $|\cdot|_{\mathcal{H}}$ the associate norm. The mapping $\mathbf{1}_{[0,t]} \mapsto B_t$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space $H_1(B)$ associated with B. We denote this isometry by $\varphi \mapsto B(\varphi)$.

When $H \in (\frac{1}{2}, 1)$, it follows from [15] that the elements of $\mathcal{H}$ may be not functions but distributions of negative order. It will be more convenient to work with a subspace of $\mathcal{H}$, which contains only functions. Such a space is the set $|\mathcal{H}|$ of all measurable functions f on $[0,T]$ such that

$$|f|^2_{|\mathcal{H}|} := H(2H-1)\int_0^T\int_0^T |f(u)||f(v)||u-v|^{2H-2}dudv < \infty.$$

We know that $(|\mathcal{H}|, |\cdot|_{|\mathcal{H}|})$ is a Banach space but that $(|\mathcal{H}|, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ is not complete (see *e.g.* [15]).

Moreover, we have the inclusions

$$\mathrm{L}^2([0,T]) \subset \mathrm{L}^{\frac{1}{H}}([0,T]) \subset |\mathcal{H}| \subset \mathcal{H}. \tag{2}$$

2.2. *Fractional operators.* The covariance kernel $R_H(t,s)$ introduced in (1) can be written as

$$R_H(t,s) = \int_0^{s\wedge t} K_H(s,u)K_H(t,u)du,$$

where $K_H(t,s)$ is the square integrable kernel defined by

$$K_H(t,s) = c_H\, s^{\frac{1}{2}-H}\int_s^t (u-s)^{H-\frac{3}{2}}u^{H-\frac{1}{2}}du, \quad 0<s<t, \tag{3}$$

where ${c_H}^2 = H(2H-1)\beta(2-2H, H-1/2)^{-1}$ and β denotes the Beta function. By convention, we set $K_H(t,s) = 0$ if $s \geq t$.

Let $\mathcal{K}_H^* : \mathcal{E} \to \mathrm{L}^2([0,T])$ be the linear operator defined by:

$$\mathcal{K}_H^*\left(\mathbf{1}_{[0,t]}\right) = K_H(t,\cdot).$$

The following equality holds for any $\phi, \psi \in \mathcal{E}$

$$\langle \phi,\psi\rangle_{\mathcal{H}} = \langle \mathcal{K}_H^*\phi, \mathcal{K}_H^*\psi\rangle_{\mathrm{L}^2([0,T])} = \mathrm{E}\left(B(\phi)B(\psi)\right)$$

and then $\mathcal{K}_H^*$ provides an isometry between the Hilbert spaces $\mathcal{H}$ and a closed subspace of $\mathrm{L}^2([0,T])$.

The process $W = (W_t)_{t\in[0,T]}$ defined by

$$W_t = B\big((\mathcal{K}_H^*)^{-1}(\mathbf{1}_{[0,t]})\big)$$

is a Wiener process, and the process B has an integral representation of the form

$$B_t = \int_0^t K_H(t,s)dW_s.$$

Hence, for any $\phi \in \mathcal{H}$,

$$B(\phi) = W\left(\mathcal{K}_H^*\phi\right).$$

Let $a, b \in \mathbb{R}$, $a < b$. For any $p \geqslant 1$, we denote by $\mathrm{L}^p = \mathrm{L}^p([a,b])$ the usual Lebesgue space of functions on $[a,b]$.

Let $f \in \mathrm{L}^1$ and $a > 0$. The left-sided and right-sided fractional Riemann-Liouville integrals of f of order α are defined for almost all $x \in (a,b)$ by

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y)dy,$$

and

$$I_{b-}^{\alpha} f(x) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y)dy,$$

respectively, where Γ denotes the usual Euler function. These integrals extend the classical integral of f when $\alpha = 1$.

If $f \in I_{a+}^{\alpha}(\mathrm{L}^p)$ (resp. $f \in I_{b-}^{\alpha}(\mathrm{L}^p)$) and $\alpha \in (0,1)$, then for almost all $x \in (a,b)$, the left-sided and right-sided Riemann-Liouville derivative of f of order α are defined by

$$D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x)-f(y)}{(x-y)^{\alpha+1}} dy \right)$$

and

$$D_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(b-x)^{\alpha}} + \alpha \int_x^b \frac{f(x)-f(y)}{(y-x)^{\alpha+1}} dy \right)$$

respectively, where $a \leq x \leq b$.

We define the operator $\mathcal{K}_H$ on $\mathrm{L}^2([0,T])$ by

$$(\mathcal{K}_H h)(t) = \int_0^t K_H(t,s)h(s)ds.$$

It is an isomorphism from $\mathrm{L}^2([0,T])$ onto $I_{0^+}^{H+\frac{1}{2}}(\mathrm{L}^2([0,T]))$ and it can be expressed as follows when $H > \frac{1}{2}$:

$$\mathcal{K}_H h = I_{0^+}^1 s^{H-\frac{1}{2}} I_{0^+}^{H-\frac{1}{2}} s^{\frac{1}{2}-H} h$$

where $h \in \mathrm{L}^2([0,T])$. The crucial point is that the functions of the space $I_{0+}^{H+\frac{1}{2}}(\mathrm{L}^2([0,T]))$ are absolutely continuous when $H > \frac{1}{2}$. For these functions ϕ, the inverse operator $\mathcal{K}_H^{-1}$ is given by

$$\mathcal{K}_H^{-1}\phi = s^{H-\frac{1}{2}} D_{0+}^{H-\frac{1}{2}} s^{\frac{1}{2}-H}\phi'.$$

When $H > \frac{1}{2}$, we introduce the operator $\mathcal{O}_H$ on $\mathrm{L}^2([0,T])$ defined by

$$(\mathcal{O}_H\varphi)(s) := \left(\frac{d}{dt}\mathcal{K}_H\right)(\varphi)(s) = s^{H-\frac{1}{2}} I_{0+}^{H-\frac{1}{2}} s^{\frac{1}{2}-H}\varphi(s). \tag{4}$$

Let $f : [a,b] \to \mathbb{R}$ be α-Hölder continuous and $g : [a,b] \to \mathbb{R}$ be β-Hölder continuous with $\alpha + \beta > 1$. Then, for any $s,t \in [a,b]$, the Young integral [20] $\int_s^t f dg$ exists and we can express it in terms of fractional derivatives (see [21]): for any $\gamma \in (1-\beta, \alpha)$, we have

$$\int_s^t f dg = (-1)^\gamma \int_s^t D_{s+}^\gamma f(x) D_{t-}^{1-\gamma} g_{t-}(x)dx, \tag{5}$$

where $g_{t-}(x) = g(x) - g(t)$. In particular, we deduce that:

$$\forall s < t \in [a,b], \quad \left|\int_s^t (f(r) - f(s))dg(r)\right| \leq \kappa |f|_\alpha |g|_\beta |t-s|^{\alpha+\beta}, \tag{6}$$

where κ is a constant depending only on a, b, α and β, and if $h : [a,b] \to \mathbb{R}$ and $\mu \in (0,1]$,

$$|h|_\mu = \sup_{a \leq s < t \leq b} \frac{|h(t) - h(s)|}{|t-s|^\mu}.$$

2.3. *Malliavin calculus.* Let $\mathcal{S}$ be the set of all smooth cylindrical random variables, *i.e.* which writes $F = f(B(\phi_1), \ldots, B(\phi_n))$ where $n \geqslant 1$, $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function with compact support and $\phi_i \in \mathcal{H}$. The Malliavin derivative of F w.r.t. B is the element of $\mathrm{L}^2(\Omega, \mathcal{H})$ defined by

$$D_s^B F \ = \ \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B(\phi_1), \ldots, B(\phi_n))\phi_i(s), \quad s \in [0,T].$$

In particular $D_s^B B_t = \mathbf{1}_{[0,t]}(s)$. As usual, $\mathbb{D}^{1,2}$ denotes the closure of the set of smooth random variables with respect to the norm

$$\|F\|_{1,2}^2 \ = \ \mathrm{E}\left[F^2\right] + \mathrm{E}\left[|D_\cdot^B F|_{\mathcal{H}}^2\right].$$

The Malliavin derivative D^B verifies the chain rule: if $\varphi : \mathbb{R}^n \to \mathbb{R}$ is C_b^1 and if $(F_i)_{i=1,\dots,n}$ is a sequence of elements of $\mathbb{D}^{1,2}$ then $\varphi(F_1,\dots,F_n) \in \mathbb{D}^{1,2}$ and we have, for any $s \in [0,T]$:

$$D_s^B\,\varphi(F_1,\dots,F_n) = \sum_{i=1}^n \frac{\partial\varphi}{\partial x_i}(F_1,\dots,F_n) D_s^B F_i.$$

The divergence operator δ^B is the adjoint of the derivative operator D^B. If a random variable $u \in \mathrm{L}^2(\Omega,\mathcal{H})$ belongs to the domain of the divergence operator, that is if it verifies

$$|\mathrm{E}\langle D^B F, u\rangle_{\mathcal{H}}| \le c_u \,\|F\|_{\mathrm{L}^2} \quad \text{for any } F \in \mathcal{S},$$

then $\delta^B(u)$ is defined by the duality relationship

$$\mathrm{E}(F\delta^B(u)) = \mathrm{E}\langle D^B F, u\rangle_{\mathcal{H}},$$

for every $F \in \mathbb{D}^{1,2}$.

2.4. *Pathwise integration with respect to B.* If $X = (X_t)_{t\in[0,T]}$ and $Z = (Z_t)_{t\in[0,T]}$ are two continuous processes, we define the forward integral of Z w.r.t. X, in the sense of Russo-Vallois, by

$$\int_0^\bullet Z_s dX_s = \lim_{\varepsilon\to 0} \text{ucp} \quad \varepsilon^{-1}\int_0^\bullet Z_s(X_{s+\varepsilon} - X_s)ds, \quad t \in [0,T], \tag{7}$$

provided the limit exists. Here "ucp" means "uniform convergence in probability". If X (resp. Z) has a.s. Hölder continuous paths of order α (resp. β) with $\alpha+\beta > 1$ then $\int_0^\bullet Z_s dX_s$ exists and coincides with the usual Young integral (see [16], Proposition 2.12).

2.5. *Stochastic differential equation driven by B.* Here we assume that $H > 1/2$. We denote by C_b^k the set of all functions whose derivatives from order 1 to order k are bounded. If $\sigma \in \mathrm{C}_b^2$ and if $b \in \mathrm{C}_b^1$, then the equation

$$X_t = x_0 + \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)\,ds, \quad t \in [0,T], \tag{8}$$

admits a unique solution X in the set of processes whose paths are Hölder continuous of order $\alpha > 1 - H$. Here, the integral w.r.t. B is in the sense of Russo-Vallois, see (7). Moreover, we have a Doss-Sussmann [6, 18] type representation:

$$X_t = \phi(A_t, B_t), \quad t \in [0,T],$$

where ϕ and A are given respectively by

$$\frac{\partial \phi}{\partial x_2}(x_1,x_2)=\sigma(\phi(x_1,x_2)),\quad \phi(x_1,0)=x_1,\qquad x_1,x_2\in\mathbb{R}$$

and

$$A'_t=\exp\left(-\int_0^{B_t}\sigma'(\phi(A_t,s))ds\right)b(\phi(A_t,B_t)),\quad A_0=x_0,\qquad t\in[0,T].$$

Using this representation, we can show that X belongs to $\mathbb{D}^{1,2}$ and that

$$D_s^B X_t=\sigma(X_s)\exp\left(\int_s^t b'(X_u)du+\int_s^t \sigma'(X_u)dB_u\right)\mathbf{1}_{[0,t]}(s),\quad s,t\in[0,T].$$

(see [11], proof of Theorem B).

3. Notions related to stochastic derivatives. Let $(Z_t)_{t\in[0,T]}$ be a stochastic process defined on $(\Omega,\mathcal{F},\mathbb{P})$. In the sequel, we always assume that for any $t\in[0,T]$, $Z_t\in \mathrm{L}^2(\Omega,\mathcal{F},\mathbb{P})$. For all $t\in(0,T)$ and $h\neq 0$ such that $t+h\in(0,T)$, we set:

$$\Delta_h Z_t=\frac{Z_{t+h}-Z_t}{h}.$$

3.1. *Stochastic derivatives in a strong sense.*

Definition 1. *Set $t\in(0,T)$. We say that $\mathcal{A}^t$ (resp. $\mathcal{B}^t$) is a* forward differentiating σ-field *(resp.* backward differentiating σ-field*) for Z at t if* $\mathrm{E}[\Delta_h Z_t|\mathcal{A}^t]$ *(resp.* $\mathrm{E}[\Delta_{-h} Z_t|\mathcal{B}^t]$*) converges in probability when $h\downarrow 0^+$. In these cases, we define the so-called forward and backward derivatives*

$$D_+^{\mathcal{A}^t}Z_t = \lim_{h\downarrow 0^+}\mathrm{E}[\Delta_h Z_t\,|\mathcal{A}^t],\tag{9}$$

$$D_-^{\mathcal{B}^t}Z_t = \lim_{h\downarrow 0^+}\mathrm{E}[\Delta_{-h} Z_t\,|\mathcal{B}^t].\tag{10}$$

The set of all forward (resp. backward) differentiating σ-fields for Z at time t is denoted by $\mathcal{M}_Z^{+(t)}$ (resp. $\mathcal{M}_Z^{-(t)}$). The intuition we can have is that the more $\mathcal{M}^{\pm(t)}$ is high, the more Z is regular at time t. For instance, one has obviously that $\{\emptyset,\Omega\}\in\mathcal{M}_Z^{+(t)}$ (resp. $\in\mathcal{M}_Z^{-(t)}$) if and only if $s\mapsto \mathrm{E}(Z_s)$ is right differentiable (resp. left differentiable) at time t. At the opposite, one has that $\mathcal{F}\in\mathcal{M}_Z^{+(t)}$ (resp. $\in\mathcal{M}_Z^{-(t)}$) if and only if $s\mapsto Z_s$ is a.s. right differentiable (resp. left differentiable) at time t.

Definition 2. *We say that* $(\mathcal{A}^t, \mathcal{B}^t)_{t\in(0,T)}$ *is a* differentiating collection of σ-fields *for* Z *if for any* $t \in (0,T)$, $\mathcal{A}^t$ *(resp.* $\mathcal{B}^t$*) is a forward (resp. backward) differentiating* σ*-field for* Z *at* t*. If* $\mathcal{A}^t = \mathcal{B}^t$ *for any* $t \in (0,T)$*, we write, for simplicity,* $(\mathcal{A}^t)_{t\in(0,T)}$ *instead of* $(\mathcal{A}^t, \mathcal{B}^t)_{t\in(0,T)}$.

On one hand, we may introduce the following:

Definition 3. *Set* $t \in (0,T)$*. We say that* $\mathcal{A}^t$ *(resp.* $\mathcal{B}^t$*) is a* non degenerated forward σ-field *(resp.* non degenerated backward σ-field*) for* Z *at* t *if it is forward (resp. backward) differentiating at* t *and if*

$$\text{for any } c \in \mathbb{R},\ \mathrm{P}(D_+^{\mathcal{A}^t} Z_t = c) = 0 \ \text{(resp. } \mathrm{P}(D_-^{\mathcal{B}^t} Z_t = c) = 0\text{)}. \tag{11}$$

For instance, if Z is a process such that $s \mapsto \mathrm{E}(Z_s)$ is differentiable at $t \in (0,T)$ then $\{\emptyset, \Omega\}$ is a forward and backward differentiating σ-field at t but is degenerated. Let us also note that the condition (11) is obviously equivalent to $\mathrm{Var}(D_+^{\mathcal{A}^t} Z_t) \neq 0$ (resp. $\mathrm{Var}(D_-^{\mathcal{B}^t} Z_t) \neq 0$) when $D_+^{\mathcal{A}^t} Z_t \in \mathrm{L}^2(\Omega)$ (resp. $D_-^{\mathcal{B}^t} Z_t \in \mathrm{L}^2(\Omega)$).

On the other hand, one could hope that such stochastic derivatives conserve the property which holds for ordinary derivatives on functions: "it can discriminate the constants among the other processes". So we introduce:

Definition 4. *We say that* $(\mathcal{A}^t, \mathcal{B}^t)_{t\in(0,T)}$ *is a* discriminating collection of σ-fields *for* Z *if* $(\mathcal{A}^t, \mathcal{B}^t)_{t\in(0,T)}$ *is a differentiating collection of* σ*-fields for* Z *and if it satisfies the following property:*

$$\Big(\forall t \in (0,T),\ D_+^{\mathcal{A}^t} Z_t = D_-^{\mathcal{B}^t} Z_t = 0\Big) \Rightarrow Z \text{ is a.s. a constant process on [0,T].}$$

As in Definition 2, we write, for simplicity, $(\mathcal{A}^t)_{t\in(0,T)}$ *instead of* $(\mathcal{A}^t, \mathcal{B}^t)_{t\in(0,T)}$ *when* $\mathcal{A}^t = \mathcal{B}^t$ *for any* $t \in (0,T)$.

An obvious example of discriminating collection of σ-fields for a process with differentiable paths is $\{\mathcal{A}^t = \mathcal{F},\ t \in (0,T)\}$. If Z is a process such that $s \mapsto \mathrm{E}(Z_s)$ is differentiable on $(0,T)$ then the collection $\{\mathcal{A}^t = \{\emptyset, \Omega\},\ t \in [0,T]\}$ is differentiating but, in general, not discriminating.

Let us now consider a more advanced example. Let $B = (B_t)_{t\in[0,T]}$ be a fractional Brownian motion with Hurst index $H \in (1/2, 1)$. Let us denote by $\mathcal{P}_t$ the σ-field generated by B_s for $0 \leq s \leq t$ and, if $g : \mathbb{R} \to \mathbb{R}$, by $\mathcal{T}_t^g$ the σ-field generated by $g(B_t)$.

Example 5. *For any $t \in (0,T)$, $\mathcal{P}_t$ is not a forward differentiating σ-field for B at t.*

We refer to Proposition 10 in [5] for a proof. This result is extended to the case of Volterra processes in this paper, see Proposition 13.

Example 6. *For any* even *function $g : \mathbb{R} \to \mathbb{R}$ and for any $t \in (0,T)$, $\mathcal{T}_t^g$ is a forward and backward differentiating (but degenerated) σ-field for B at t.*

Proof. Since B and $-B$ have the same law, we have that $\mathrm{E}[\Delta_h B_t | g(B_t)] = 0$ for any $t \in (0,T)$ and $h \neq 0$ such that $t+h \in (0,T)$. The conclusion follows easily. $\square$

Example 7. *For any $t \in (0,T)$, $\mathcal{T}_t^{\mathrm{id}}$ is a forward and backward differentiating and non degenerated σ-field for B at t.*

Proof. Using a linear Gaussian regression, we can write

$$\mathrm{E}[\Delta_h B_t | B_t] = \frac{(1+h/t)^{2H} - 1 - (h/t)^{2H}}{2} B_t \xrightarrow[h \to 0]{} H \frac{B_t}{t} \quad \text{in probability.}$$

Since $\mathrm{Var}(H\, t^{-1}\, B_t) > 0$, $\mathcal{T}_t^{\mathrm{id}}$ is non-degenerated. $\square$

Thus, for the fractional Brownian motion, stochastic derivatives w.r.t. the present (that is w.r.t. $\mathcal{T}_t^{\mathrm{id}}$) turns out to be an adequate tool (see section 5 below, for a more precise study).

3.2. *Stochastic derivatives in a weak sense.* A way to weaken Definition 1 is to consider stochastic derivatives as follows:

Definition 8. *Set $t \in (0,T)$ and let $\mathcal{A}$ be a sub-σ-field of $\mathcal{F}$. We say that Z is weak forward differentiable w.r.t. $\mathcal{A}$ at t if*

$$\lim_{h \downarrow 0^+} \mathrm{E}[V \Delta_h Z_t] \ \textit{exists},$$

for all random variable V belonging to a dense subspace of the closed subspace $\mathrm{L}^2(\Omega, \mathcal{A}, \mathbb{P}) \subset \mathrm{L}^2(\Omega, \mathcal{F}, \mathbb{P})$.
We similarly define the notion of weak backward differentiation w.r.t. $\mathcal{A}$ at t by considering $\Delta_{-h} Z_t$ instead of $\Delta_h Z_t$.

If the process Z is weak forward differentiable at t and is such that the sequence $(\Delta_h Z_t)_h$ is bounded in $\mathrm{L}^2(\Omega)$, then we can associate a weak forward stochastic derivative w.r.t. $\mathcal{A}$ at t. Indeed, in that case, let us denote by Θ the dense subspace involved. The linear form $\psi : V \mapsto \lim_{h\downarrow 0^+} \mathrm{E}[V\Delta_h Z_t]$ defined on $\Theta \subset \mathrm{L}^2(\Omega, \mathcal{A}, \mathbb{P})$ is continuous and so can be extended in a unique continuous linear form on $\mathrm{L}^2(\Omega, \mathcal{A}, \mathbb{P})$, still denoted by ψ. Thus there exists a unique $Z'_t \in \mathrm{L}^2(\Omega, \mathcal{A}, \mathbb{P})$ such that $\psi(V) = \mathrm{E}[Z'_t V]$. One can easily show that Z'_t does not depend on Θ. We will say that Z'_t is the weak forward stochastic derivative of Z w.r.t. $\mathcal{A}$ at t.

Remark 9. The boundedness of $(\Delta_h Z_t)_h$ in $\mathrm{L}^2(\Omega)$ may appear as a quite restrictive condition (for instance, it is not satisfied for a fractional Brownian motion). But it allows to relate our notion with the usual notion of weak limit.

If $\mathcal{A}^t$ (resp. $\mathcal{B}^t$) is a forward (resp. backward) differentiating σ-field for Z at t and if moreover the convergence (9) (resp. (10)) also holds in L^2, then Z is weak forward (resp. backward) differentiable w.r.t. $\mathcal{A}^t$ (resp. $\mathcal{B}^t$) at t. But the converse is not true in general.

Let Υ be the set of the so-called fractional diffusions $X = (X_t)_{t\in[0,T]}$ defined by

$$X_t = x_0 + \int_0^t \sigma_s dB_s + \int_0^t b_s ds, \quad t \in [0,T], \tag{12}$$

where σ and b are adapted w.r.t. the natural filtration associated to B and X. We moreover assume that they satisfy the following conditions: $\sigma \in C^\alpha$ a.s. with $\alpha > 1 - H$ and $b \in \mathrm{L}^1([0,T])$ a.s.

Lemma 10. *The decomposition (12) is unique: if*

$$x_0 + \int_0^t \sigma_s dB_s + \int_0^t b_s ds = \tilde{x}_0 + \int_0^t \tilde{\sigma}_s dB_s + \int_0^t \tilde{b}_s ds, \quad t \in [0,T] \tag{13}$$

then $x_0 = \tilde{x}_0$, $\sigma = \tilde{\sigma}$ and $b = \tilde{b}$.

Proof. The equality $x_0 = \tilde{x}_0$ is obvious and (13) is then equivalent to

$$\int_0^t (\sigma_s - \tilde{\sigma}_s) dB_s = \int_0^t (\tilde{b}_s - b_s) ds, \quad t \in [0,T]$$

which implies, by setting $t_k = \frac{kT}{n}$:

$$
\begin{aligned}
&\left(|\sigma_{t_k} - \tilde\sigma_{t_k}||B_{t_{k+1}} - B_{t_k}|\right)^{1/H} \\
&\quad = \left| \int_{t_k}^{t_{k+1}} (b_s - \tilde b_s) ds + \int_{t_k}^{t_{k+1}} (\sigma_s - \sigma_{t_k}) dB_s + \int_{t_k}^{t_{k+1}} (\tilde\sigma_s - \tilde{\sigma_{t_k}}) dB_s \right|^{1/H} \\
&\quad \leqslant C \left[\left| \int_{t_k}^{t_{k+1}} (b_s - \tilde b_s) ds \right|^{1/H} + \left| \int_{t_k}^{t_{k+1}} (\sigma_s - \sigma_{t_k}) dB_s \right|^{1/H} \right. \\
&\qquad\qquad \left. + \left| \int_{t_k}^{t_{k+1}} (\tilde\sigma_s - \tilde\sigma_{t_k}) dB_s \right|^{1/H} \right].
\end{aligned}
$$

We easily deduce, using (6), that

$$
\lim_{n\to\infty} \sum_{k=0}^{n-1} |\sigma_{t_k} - \tilde\sigma_{t_k}|^{1/H} |B_{t_{k+1}} - B_{t_k}|^{1/H} = 0 \quad \text{in probability.}
$$

But, on the other hand, it is easy to obtain (see, for instance, Theorem 4.4 in [8]) that:

$$
\lim_{n\to\infty} \sum_{k=0}^{n-1} |\sigma_{t_k} - \tilde\sigma_{t_k}|^{1/H} |B_{t_{k+1}} - B_{t_k}|^{1/H} = \int_0^T |\sigma_s - \tilde\sigma_s|^{1/H} ds \text{ in probability.}
$$

We deduce that $\sigma = \tilde\sigma$ and then $b = \tilde b$. □

In section 4, we will see that the past of $X \in \Upsilon$ before time t is not, in general, a forward differentiating σ-field at time t. At the opposite, we will see in section 5 that the present of $X \in \Upsilon$ is, in general, a differentiating collection of σ-fields.

However, X is weak differentiable for a large class of σ-fields. We introduce the set $\mathcal{S}^b$ of all r.v. $\varphi(B(\phi_1), \cdots, B(\phi_n)) \in \mathcal{S}$ such that $\phi_1, \cdots, \phi_n$ are bounded functions.

Let $\wp$ be the set of all sub-σ-fields $\mathcal{A} \subset \mathcal{F}$ such that $\mathrm{L}^2(\Omega, \mathcal{A}, \mathbb{P}) \cap \mathcal{S}^b$ is dense in $\mathrm{L}^2(\Omega, \mathcal{A}, \mathbb{P})$. For instance, any σ-field which writes $\mathcal{A}^{[r,s]} = \varsigma(B_v, r \leqslant v \leqslant s)$ belongs to $\wp$ (see *e.g.* [13] p.24).

PROPOSITION 11. *Let $\mathcal{A} \in \wp$ and $t \in (0, T)$. Let $X \in \Upsilon$ be given by (12) satisfying the following conditions:*

(i) The map $s \mapsto b_s$ is continuous from $(0, T)$ into $L^1(\Omega)$,
(ii) for all $s \in [0, T]$, $\sigma_s \in \mathbb{D}^{1,2}$ and $\sup_{s\in[0,T]} \mathrm{E}|D_s^B \sigma_t| < +\infty$,

(iii) $\mathrm{E}(|\sigma|_\alpha^p) < +\infty$ *for some* $p > 1$ *and* $\alpha > 1 - H$.

Then X *is weak forward and backward differentiable at* t *w.r.t.* $\mathcal{A}$.

PROOF. For simplicity, we only prove the forward case, the backward case being similar. Let $t \in (0, T)$.

We write

$$X_{t+h} - X_t = \sigma_t(B_{t+h} - B_t) + \int_t^{t+h} b_s ds + \int_t^{t+h} (\sigma_s - \sigma_t) dB_s. \tag{14}$$

First of all, we treat the second term of the r.h.s. of (14). Let $V \in \mathrm{L}^2(\Omega, \mathcal{A}, \mathbb{P}) \cap \mathcal{S}^b$. Since V is bounded and the map $s \mapsto b_s$ is continuous from $(0, T)$ into $L^1(\Omega)$, the function $s \mapsto \mathrm{E}\left[V b_s\right]$ is continuous. We then deduce, together with the Fubini theorem, that

$$\lim_{h \downarrow 0} \frac{1}{h} \mathrm{E}\left[V \int_t^{t+h} b_s\, ds\right] ds = \mathrm{E}\left[V b_t\right]. \tag{15}$$

Afterwards, using the inequality (6) and the hypothesis $\mathrm{E}(|\sigma|_\alpha^p) < +\infty$, the following limit holds:

$$\lim_{h \to 0} \frac{1}{h}\, \mathrm{E}\left[V \int_t^{t+h} (\sigma_s - \sigma_t) dB_s\right] = 0. \tag{16}$$

Finally, we show that the limit

$$\lim_{h \downarrow 0} \mathrm{E}[\sigma_t\, V\, \Delta_h B_t]$$

exists. Since $\sigma_t V \in \mathbb{D}^{1,2}$ (see Exercice 1.2.13 in [12]), we have

$$\begin{aligned} \mathrm{E}[\sigma_t(B_{t+h} - B_t)V] &= \mathrm{E}[\delta^B(\mathbf{1}_{[t,t+h]})\, \sigma_t\, V] \\ &= \mathrm{E}[\sigma_t\, \langle \mathbf{1}_{[t,t+h]}, D^B V\rangle_{\mathcal{H}}] + \mathrm{E}\left[V \langle \mathbf{1}_{[t,t+h]}, D^B \sigma_t\rangle_{\mathcal{H}}\right]. \end{aligned}$$

The condition (ii) and the fact that $V \in \mathcal{S}^b$ allow in particulary to write

$$\mathrm{E}[\sigma_t(B_{t+h} - B_t)V] = H(2H-1)(\Psi_{t,h}(\sigma_t, V) + \Psi_{t,h}(V, \sigma_t)) \tag{17}$$

where

$$\Psi_{t,h}(X, Y) = \mathrm{E}\left[X \int_0^T D_s^B Y \int_t^{t+h} |v - s|^{2H-2} dv ds\right].$$

When X or Y denotes σ_t or V, the Fubini theorem yields to

$$\Psi_{t,h}(X, Y) = \int_t^{t+h} f(v, X, Y) dv,$$

with

$$f(v,X,Y)=\int_0^T \mathrm{E}[XD_s^B Y]|v-s|^{2H-2}ds.$$

We have thanks to the condition (ii) and the fact that $V\in\mathcal{S}^b$ again:

$$|f(v,X,Y)-f(w,X,Y)|\leqslant C(X,Y)\int_0^T \left||v-s|^{2H-2}-|w-s|^{2H-2}\right|ds,$$

where $C(X,Y)$ is a constant depending only on X and Y. This inequality shows the continuity of the function $v\mapsto f(v,X,Y)$ thanks to a straightforward study of the involved integral.

Therefore, the limit

$$\lim_{h\to 0} h^{-1}\,\mathrm{E}[\sigma_t(B_{t+h}-B_t)V]$$

exists and equals

$$H(2H-1)\mathrm{E}\left[\sigma_t\int_0^T D_s^B V|t-s|^{2H-2}ds+V\int_0^T D_s^B\sigma_t|t-s|^{2H-2}ds\right].$$

□

4. Stochastic derivatives of Nelson's type. Let Z be a stochastic process defined on $(\Omega,\mathcal{F},\mathbb{P})$. We define the past of Z before time t:

$$\mathcal{P}_t^Z:=\varsigma(Z_s,\,0\le s\le t)$$

and the future of Z after time t:

$$\mathcal{F}_t^Z:=\varsigma(Z_s,\,t\le s\le T).$$

If $\mathcal{P}_t^Z$ and $\mathcal{F}_t^Z$ are respectively forward and backward differentiating σ-fields for Z at t, we call $D_+^{\mathcal{P}_t^Z}Z_t$ and $D_-^{\mathcal{F}_t^Z}Z_t$ respectively the forward and backward stochastic derivatives of Nelson's type in reference of Nelson's work [10]. In the sequel, we denote them by $D_+^{\mathcal{P}}Z_t$ and $D_-^{\mathcal{F}}Z_t$ for simplicity.

4.1. *The case of Wiener diffusions.* We denote by Λ the space of diffusion processes X satisfying the following conditions:

1. X solves the stochastic differential equation :

$$dX_t=b(t,X_t)dt+\sigma(t,X_t)dW_t,\quad X_0=x_0, \tag{18}$$

where $x_0 \in \mathbb{R}^d$, $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ are Borel measurable functions satisfying the hypothesis : there exists a constant $K > 0$ such that for every $x, y \in \mathbb{R}^d$ we have

$$\sup_t \left(|\sigma(t,x) - \sigma(t,y)| + |b(t,x) - b(t,y)|\right) \leqslant K\,|x-y|\,,$$
$$\sup_t \left(|\sigma(t,x)| + |b(t,x)|\right) \leqslant K(1+|x|).$$

2. For any $t \in (0,T)$, X_t has a density p_t.
3. Setting $a_{ij} = (\sigma\sigma^*)_{ij}$, for any $i,j \in \{1,\cdots,n\}$, for any $t_0 \in (0,T)$, for any bounded open set $O \subset \mathbb{R}^d$,

$$\int_{t_0}^T \int_O |\partial_j(a_{ij}(t,x)p_t(x))|\,dxdt < +\infty.$$

4. The functions b and $(t,x) \mapsto \dfrac{1}{p_t(x)}\partial_j(a_{ij}(t,x)p_t(x))$ are bounded, belong to $C^{1,2}([0,T]\times\mathbb{R}^d)$, and have all its first and second order derivatives bounded (we use the usual convention that the term involving $\frac{1}{p_t(x)}$ is 0 if $p_t(x) = 0$).

These conditions are introduced in [9] and ensure the existence of a drift and a diffusion coefficient for the time reversed process $\overline{X}_t := X_{T-t}$. Föllmer focuses in [7] Proposition 2.5 on the important relation between drifts and derivatives of Nelson's type. It allows him to compute the drift of the time reversal of a Brownian diffusion with a constant diffusion coefficient both in the Markov and non Markov case (see Theorem 3.10 and 4.7 in [7]).

For a Markov diffusion with a rather general diffusion coefficient, we have the following

THEOREM 12. *Let $X \in \Lambda$ given by (18). Then X is a Markov diffusion with respect to $\mathcal{P}^X$ and $\mathcal{F}^X$. Moreover, $\mathcal{P}^X$ and $\mathcal{F}^X$ are differentiating and, in general, non degenerated:*

$$\begin{aligned} D_+^{\mathcal{P}} X_t &= b(t,X_t), \\ (D_-^{\mathcal{F}} X_t)_i &= b_i(t,X_t) - \frac{1}{p_t(X_t)} \sum_j \partial_j(a_{ij}(t,X_t)p_t(X_t)), \end{aligned}$$

where the convention that the term involving $\frac{1}{p_t(x)}$ is 0 if $p_t(x) = 0$.

We refer to [4] for a proof: it is based on the proof of Proposition 4.1 in [19] and Theorem 2.3 in [9].

4.2. *The case of fractional Brownian motion and Volterra processes.* Let K be an L^2-kernel, that is a function $K : [0,T] \times [0,T] \to \mathbb{R}$ verifying $\int_{[0,T]^2} K(t,s)^2 dtds < +\infty$. We denote by $\frac{\partial^+ K}{\partial t}$ the *right* derivative of K with respect to t (with the convention that it equals to $+\infty$ if it does not exist).

We assume moreover that K is Volterra: that is it vanishes on $\{(t,s) \in [0,T]^2 : s > t\}$, and is non degenerated: that is the family $\{K(t,\cdot),\ t \in [0,T]\}$ is free and span a vector space dense in $\mathrm{L}^2([0,T])$. For such a kernel K, we associate the so-called Volterra process

$$G_t = \int_0^t K(t,s) dW_s, \quad 0 \le t \le T, \tag{19}$$

where W denotes a standard Brownian motion. The assumptions made on K imply in particular that the natural filtrations associated to W and G are the same (see for instance [2], Remark 3).

Proposition 13. *Let $t \in (0,T)$ and G be a Volterra process associated to a non degenerated Volterra kernel K satisfying the condition:*

$$\frac{K(t+h,\cdot) - K(t,\cdot)}{h} \underset{h\downarrow 0}{\longrightarrow} \frac{\partial^+ K}{\partial t} \ \textit{in } \mathrm{L}^2([0,t]). \tag{20}$$

The forward Nelson derivative $D_+^{\mathcal{P}} G_t$ at t exists if and only if $\int_0^t \frac{\partial^+ K}{\partial t}(t,s)^2 ds < +\infty$. In this case, we have $D_+^{\mathcal{P}} G_t = \int_0^t \frac{\partial^+ K}{\partial t}(t,s) dW_s$ and $\mathcal{P}_t^G$ is non degenerated at t if and only if $\int_0^t \frac{\partial^+ K}{\partial t}(t,s)^2 ds > 0$.

Proof. We adapt the proof of [5], Proposition 10. Using the representation (19), we deduce that

$$\begin{aligned} \mathrm{E}\left[\Delta_h G_t \,|\mathcal{P}_t^G\right] &= \mathrm{E}\left[\Delta_h G_t \,|\mathcal{P}_t^W\right] \\ &= \frac{1}{h} \int_0^t [K(t+h,s) - K(t,s)] dW_s =: Z_h. \end{aligned}$$

Remark that $Z = (Z_h)_{h>0}$ is a centered Gaussian process. First assume that $\int_0^t \frac{\partial^+ K}{\partial t}(t,s)^2 ds = +\infty$. It is classical that, if Z_h converges in probability as $h \downarrow 0$, then $\mathrm{Var}(Z_h)$ converges as $h \downarrow 0$. But, from Fatou's lemma, we deduce

$$\liminf_{h\downarrow 0}\ \mathrm{Var}(Z_h) \ge \int_0^t \frac{\partial^+ K}{\partial t}(t,s)^2 ds = +\infty.$$

Thus, Z_h does not converge in probability as $h \downarrow 0$. Conversely, assume that $\int_0^t \frac{\partial^+ K}{\partial t}(t,s)^2 ds < +\infty$. In this case, the assumption (20) implies that

$Z_h \to \int_0^t \frac{\partial^+ K}{\partial t}(t,s)dW_s$ in probability, as $h \downarrow 0$. In other words, $D_+^{\mathcal{P}} G_t$ exists and equals $\int_0^t \frac{\partial^+ K}{\partial t}(t,s)dW_s$. We easily deduce that $\mathcal{P}_t^G$ is non degenerated at t if and only if $\int_0^t \frac{\partial^+ K}{\partial t}(t,s)^2 ds > 0$. $\square$

The result of Proposition 10 in [5] is then a particular case: if B denotes a fractional Brownian motion with Hurst index $H \in (0,1/2) \cup (1/2,1)$ and if $t \in (0,T)$, then $D_+^{\mathcal{P}} B_t$ does not exist. Indeed, we have $B_t = \int_0^t K_H(t,s)dW_s$ where K_H is the non-degenerated Volterra kernel given by (3) and verifying

$$\frac{\partial K_H}{\partial t}(t,s) = c_H \left(\frac{t}{s}\right)^{H-1/2} (t-s)^{H-3/2}.$$

Remark 14. For a stochastic process Z, let us define

$$\xi(Z) = \mathrm{Leb}\{t \in [0,T],\ D_+^{\mathcal{P}} Z_t \text{ exists}\}. \tag{21}$$

For instance, if B is a fractional Brownian motion with Hurst index $H \in (0,1)$, then $\xi(B) = T$ if $H = 1/2$ and $\xi(B) = 0$ otherwise. A real $c \in [0,T]$ being fixed, it is in fact not difficult, by using Proposition 13, to construct a continuous process Z such that $\xi(Z) = c$. For instance, we can consider the Volterra process associated to the Volterra kernel

$$K(t,s) = \begin{cases} (t-s)^{H(t)} & \text{if } s \le t \\ 0 & \text{otherwise} \end{cases} \quad \text{with } H(t) = \begin{cases} 0 & \text{if } t \le c \\ (t-c) \wedge 1/4 & \text{if } t > c \end{cases}.$$

The study of backward derivatives seems to be more difficult. Among these difficulties, we mention the fact that it is not easy to obtain backward representation of fractional diffusions (see [5]). However, for a fractional Brownian motion, we are able to prove the following proposition:

Proposition 15. *Set $H > 1/2$. The limit*

$$\lim_{h \downarrow 0} \mathrm{E}\left[\left.\frac{B_t - B_{t-h}}{h}\right| \mathcal{F}_t^B\right]$$

exists neither as an element in $\mathrm{L}^p(\Omega)$ for any $p \in [1,\infty)$ nor as an almost sure limit.

Proof. We fix $t \in (0,T)$. We set

$$G_h := \mathrm{E}\left[\left.\frac{B_t - B_{t-h}}{h}\right| \mathcal{F}_t^B\right] \quad \text{and} \quad Z_h := \mathrm{E}\left[\left.\frac{B_t - B_{t-h}}{h}\right| B_t, B_{t+h}\right].$$

Since $(G_h)_{h>0}$ is a family of Gaussian random variables, it only suffices to prove that $\mathrm{Var}(G_h)$ diverges when h goes to 0.

We have : $Z_h = \mathrm{E}\left[G_h | B_t, B_{t+h}\right]$. So, by Jensen inequality, $Z_h^2 \leqslant \mathrm{E}\left[G_h^2 | B_t, B_{t+h}\right]$ and $\mathrm{Var}(Z_h) \leqslant \mathrm{Var}(G_h)$. Let us show that $\lim_{h\downarrow 0} \mathrm{Var}(Z_h) = +\infty$.

The covariance matrix of the Gaussian vector $(B_{t-h} - B_t, B_t, B_{t+h})$ reads

$$\left(\begin{array}{cc} a & v \\ v^* & M \end{array} \right)$$

where $a = \mathrm{Var}(B_{t-h} - B_t)$, $v = (R(t-h,t) - R(t,t)\,; R(t-h,t+h) - R(t,t+h))$ and

$$M = \left(\begin{array}{cc} R(t,t) & R(t,t+h) \\ R(t,t+h) & R(t+h,t+h) \end{array} \right).$$

Since $d_h := R(t,t)R(t+h,t+h) - R(t+h,t)^2 \neq 0$, M is invertible. Therefore $hZ_h = vM^{-1}Q^*$ where $Q = (B_t, B_{t+h})$. Since $M = \mathrm{E}[Q^*Q]$, we deduce that

$$\mathrm{Var}(hZ_h) = \mathrm{E}[vM^{-1}Q^*(vM^{-1}Q^*)^*] = vM^{-1}v^*.$$

Hence

$$\mathrm{Var}(hZ_h) = \frac{1}{d_h}\left(R(t+h,t+h)v_1^2 - 2R(t+h,t)v_1v_2 + R(t,t)v_2^2 \right).$$

This expression is homogeneous in t^{2H}, so we henceforth work with $t = 1$. Tedious computations give $d_h \sim h^{2H}$ as $h \downarrow 0$. Moreover we note that $v_2 = v_1 + \mathrm{c}\, h^{2H}$ where c is a constant depending only on H. Thus

$$d_h \, \mathrm{Var}(hZ_h) = v_1 \, \mathrm{c}\, h^{2H}(1 - (1+h)^{2H} + h^{2H}) + h^{2H} v_1^2 + \mathrm{c}^2\, h^{4h}.$$

Since $2H > 1$ and the function $x \mapsto x^{2H}$ is derivable, the quantities $\frac{v_1}{h}$ and $\frac{1-(1+h)^{2H}+h^{2H}}{h}$ converge as $h \downarrow 0$. But $2H < 2$ and $\frac{h^{4H}}{h^2 h^{2H}} = h^{2H-2} \to +\infty$ as $h \downarrow 0$. Thus

$$\lim_{h\downarrow 0} \mathrm{Var}(Z_h) = +\infty,$$

which concludes the proof.

□

4.3. *The case of fractional diffusions.*

PROPOSITION 16. *Let $X \in \Upsilon$ given by (12) and satisfying the following conditions:* $\mathrm{E}\left(\int_0^T |b_s| ds\right) < +\infty$ *and* $\mathrm{E}(|\sigma|_\alpha^p) < +\infty$ *for some $p > 1$ and $\alpha > 1 - H$. If, for any $t \in (0,T)$, $\sigma_t \neq 0$ a.s. then for almost all $t \in (0,T)$, $\mathcal{P}_t^X$ is not a forward differentiating σ-field for X at t.*

PROOF. Remember we assumed that σ and b are adapted w.r.t. the natural filtration associated to B and X, see (12). In particular, we deduce from (12) that $\mathcal{P}_t^X \subset \mathcal{P}_t^B$. Since we can also write

$$B_t = \int_0^t \frac{1}{\sigma_s}\, dX_s - \int_0^t \frac{b_s}{\sigma_s}\, ds,$$

we finally have $\mathcal{P}_t^X = \mathcal{P}_t^B$.

Thus, we deduce that $\mathrm{E}[\Delta_h B_t \,|\, \mathcal{P}_t^X] = \mathrm{E}[\Delta_h B_t \,|\, \mathcal{P}_t^B]$ does not converge in probability as $h \downarrow 0$, as a consequence of Proposition 10 in [5] or Proposition 13 of this paper.

Consider the expression (14). The hypothesis $E\int_0^T |b_s| ds < +\infty$ allows us to use the techniques of the proof of Proposition 2.5 in [7] to show that $\frac{1}{h}E[\int_t^{t+h} b_s ds | \mathcal{P}_t^X]$ converges in probability for almost all t. Using now the inequality (6) and the hypothesis $\mathrm{E}(|\sigma|_\alpha^p) < +\infty$, we can finally conclude that $\mathcal{P}_t^X$ is not a forward differentiating σ-field for X at almost all time t. □

4.4. *The case of fractional differential equations with analytic volatility.*

PROPOSITION 17. *Let $X \in \Xi$ given by (8) and $t \in (0,T)$. We assume moreover that σ is a real analytic function. Then $\mathcal{P}_t^X$ is a forward differentiating σ-field for X at t if and only if $\sigma \equiv 0$. In this case, $\mathcal{P}^X = \{\mathcal{P}_t^X, t \in (0,T)\}$ is a discriminating collection of σ-fields and $\mathcal{P}_t^X$ is degenerated at any $t \in (0,T)$.*

PROOF. It $\sigma \equiv 0$ then X is deterministic, and differentiable in t. Consequently, $\mathcal{P}_t^X$ is a forward differentiating σ-field but is degenerated. Assume now that $\sigma \not\equiv 0$. According to the Bouleau-Hirsch optimal criterium for fractional differential equations (see [11], Theorem B), we have that the law of X_t is absolutely continuous w.r.t. the Lebesgue measure for any t (we have indeed $\mathrm{int}\,\sigma^{-1}(\{0\}) = \emptyset$). We deduce that $\mathrm{P}(\sigma(X_t) = 0) = 0$ for any t, since $\mathrm{Leb}(\sigma^{-1}(\{0\})) = 0$ (σ has only isolated zeros). Proposition 16 allows to conclude that $\mathcal{P}_t^X$ is not a forward differentiating σ-field. □

REMARK 18. The case where σ is not assumed analytical seems more difficult to reach. We conjecture however that, in this case, $\mathcal{P}_t^X$ is a forward differentiating σ-field for X if and only if $t < t_x$ where t_x is the deterministic time defined by

$$t_x = \inf\{t \geq 0 :\, x_t \notin \mathrm{int}\,\sigma^{-1}(\{0\})\}$$

with $(x_t)_{t\in[0,T]}$ the solution to $x_t = x_0 + \int_0^t b(x_s)ds$. If this conjecture is true, we would have that $\xi(X) = t_x$, see (21).

5. Stochastic derivatives with respect to the present.

5.1. *Definition.* A consequence of Proposition 13 is that the σ-field $\mathcal{P}_t^X$ generated by X_s, $0 \le s \le t$ (the past of X) is not an adequate objet as regards our differentiation when we work with the fractional Brownian motion. Moreover, we can stress on the following important fact: the Markov property of a Wiener diffusion $X \in \Lambda_d$ implies that to take expectations w.r.t. $\mathcal{P}_t^X$ produces the same effect as to take expectations only w.r.t. X_t. The following definition is then natural.

DEFINITION 19. *Let $Z = (Z_t)_{t\in[0,T]}$ be a stochastic process defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and, for any $t \in (0,T)$, $\mathcal{T}_t^Z$ be the σ-field generated by Z_t. We say that Z admits a forward (resp. backward) stochastic derivative w.r.t. the present $t \in (0,T)$ if $\mathcal{T}_t^Z$ is a forward (resp. backward) differentiating σ-field for Z at t. In this case, we set $D_+^{\mathcal{T}} Z_t := D_+^{\mathcal{T}_t^Z} Z_t$ (resp. $D_-^{\mathcal{T}} Z_t := D_-^{\mathcal{T}_t^Z} Z_t$).*

EXAMPLE 20. Let B be a fractional Brownian motion with Hurst index $H \in (0,1)$ and $t \in (0,T)$. Then

$$D_+^{\mathcal{T}} B_t = \begin{cases} H\, t^{-1}\, B_t & \text{if } H > 1/2 \\ 0 & \text{if } H = 1/2 \\ \text{does not exist} & \text{if } H < 1/2 \end{cases} \qquad \text{and}$$

$$D_-^{\mathcal{T}} Z_t = \begin{cases} H\, t^{-1}\, B_t & \text{if } H > 1/2 \\ t^{-1}\, B_t & \text{if } H = 1/2 \\ \text{does not exist} & \text{if } H < 1/2 \end{cases}$$

(see also Example 7). In particular, we would say that the fractional Brownian motion with Hurst index $H > 1/2$ is more regular than the Brownian motion ($H = 1/2$), because of the equality between the forward and backward derivatives in the case $H > 1/2$ contrary to the case $H = 1/2$. We can identify the cause of these different regularities: the covariance function R_H is differentiable along the diagonal (t,t) in the case $H > 1/2$ while it is not when $H = 1/2$.

5.2. *Case of fractional differential equations.* We denote by Ξ the set of fractional differential equations, that is the subset of Υ whose elements are processes $X = (X_t)_{t\in[0,T]}$ solution of (8) with $\sigma \in \mathcal{C}_b^2$ and $b \in \mathcal{C}_b^1$.

In the sequel, we compute $D_\pm^{\mathcal{T}} X_t$ for $X \in \Xi$ and $t \in (0,T)$. Let us begin by a simple case.

PROPOSITION 21. *Let $X \in \Xi$ given by (8) and $t \in (0,T)$. Assume moreover that σ and b are proportional. Then X admits a forward and a backward stochastic derivative w.r.t. the present t, given by*

$$D_+^{\mathcal{T}} X_t = D_-^{\mathcal{T}} X_t = H\, t^{-1}\sigma(X_t) B_t + b(X_t). \tag{22}$$

In particular, the present $\mathcal{T}_t^X$ is non degenerated at t if and only if $\sigma(x_0) \neq 0$ and the collection of σ-fields $\mathcal{T}^X = \{\mathcal{T}_t^X,\, t \in (0,T)\}$ is discriminating for X.

PROOF. We make only the proof for $D_+^{\mathcal{T}} X_t$, the computation for $D_-^{\mathcal{T}} X_t$ being similar. Assume that $b(x) = r\,\sigma(x)$ with $r \in \mathbb{R}$. Then $X_t = f(B_t + rt)$ with $f : \mathbb{R} \to \mathbb{R}$ defined by $f(0) = x_0$ and $f' = \sigma(f)$. If $\sigma(x_0) = 0$ then $X_t \equiv x_0$ and $D_+^{\mathcal{T}} X_t = 0 = \sigma(X_t)\, H t^{-1} B_t + b(X_t)$. If $\sigma(x_0) \neq 0$ then it is classical that f is strictly monotonous. We can then write $B_t = f^{-1}(X_t) - rt$. In particular, the random variables which are measurable with respect to X_t, are measurable with respect to B_t, and vice-versa. On the other hand, by using a linear Gaussian regression, it is easy to show that $D_+^{\mathcal{T}} B_t = H\, t^{-1} B_t$ (see also Example 7). Finally, the convergences (15) and (16) and the equality (14) allow to conclude that we have (22).

Now, let us prove that the present is non degenerated for X at t if and only if $\sigma(x_0) \neq 0$. When $\sigma(x_0) = 0$, it is clear that the present is degenerated at t (see the first part of this proof). On the other hand, if the present is degenerated at t, then there exists $c \in \mathbb{R}$ such that

$$H\, t^{-1} \sigma \circ f(B_t + rt) B_t + r\, \sigma \circ f(B_t + rt) = c.$$

By rearranging, we obtain that $\sigma \circ f(X)(X + \alpha) = \beta$ for some $\alpha, \beta \in \mathbb{R}$ and with $X = B_t + rt$. By using the fact that X has a strictly positive density on $\mathbb{R}$, we deduce that $\sigma \circ f(x)(x + \alpha) = \beta$ for any $x \in \mathbb{R}$. Necessarily, $\beta = 0$ (with $x = -\alpha$) and then $f' = \sigma \circ f = 0$. We deduce that f is constant and then that $f \equiv x_0$, that is $\sigma(x_0) = 0$.

Finally, if $H\, t^{-1}\sigma(X_t) B_t + b(X_t) = \sigma(X_t)(H\, t^{-1} B_t + r) = 0$ a.s. for any t, then $\sigma(X_t) = 0 = b(X_t)$ a.s. for any t and $X_t \equiv x_0$ a.s. for any t, see (8). In other words, the collection of σ-fields $\mathcal{T}^X = \{\mathcal{T}_t^X,\, t \in (0,T)\}$ is discriminating. $\square$

Let us now describe a more general case.

THEOREM 22. *Let $X \in \Xi$ given by (8) and $t \in (0,T)$. Assume moreover that $b \in \mathcal{C}_b^2$ and that $\sigma \in \mathcal{C}_b^2$ is elliptic, that is verifies $\inf_{x \in \mathbb{R}} |\sigma(x)| > 0$.*

Then X admits a forward and a backward stochastic derivative w.r.t. the present t, given by

$$
\begin{aligned}
D_+^{\mathcal{T}} X_t &= D_-^{\mathcal{T}} X_t \\
&= b(X_t) + H\,\frac{\sigma(X_t)}{t}\left\{\int_0^{X_t}\frac{dy}{\sigma(y)}\right. \\
&\left. -\mathrm{E}\left[\int_0^t \frac{b}{\sigma}(X_s)ds + \int_0^t\int_0^t \beta_r^H(s)\delta B_r ds - t\int_0^t \beta_r^H(t)\delta B_r \,\middle|\, X_t\right]\right\}
\end{aligned}
\tag{23}
$$

where

$$\beta_r^H(t) = \left(\mathcal{O}_H \int_0^r \frac{b'\sigma - b\sigma'}{\sigma}(X_s)\mathbf{1}_{s\geq\cdot}\, ds\right)(t).$$

Recall that $\mathcal{O}_H$ is defined by (4).

Proof. Remark first that $\beta_r^H(t)$ belongs to the domain of the divergence operator δ^B, due to the additional hypothesis on b and σ. We only make the proof for $D_+^{\mathcal{T}} X_t$, the computation for $D_-^{\mathcal{T}} X_t$ being similar.

First step. Assume that $\sigma \equiv 1$. Using the transfer principle and the isometry $\mathcal{K}_H$, it holds that

$$X_t = \int_0^t K_H(t,s)dY_s$$

where

$$Y_t = W_t + \int_0^t a_r dr.$$

Here, we set

$$a_r = \left(\mathcal{K}_H^{-1}\int_0^{\cdot} b(X_s)ds\right)(r).$$

We know (see [14], Theorem 2) that the process $X = (X_t)_{t\in[0,T]}$ is a fractional Brownian motion under the new probability measure $\mathbb{Q} = G\cdot\mathbb{P}$ where

$$G = \exp\left(-\int_0^T a_s dW_s - \frac{1}{2}\int_0^T a_s^2 ds\right).$$

Using the integration by part of Malliavin calculus, we can write, for $g : \mathbb{R}\to\mathbb{R}\in \mathrm{C}_b^1$:

$$
\begin{aligned}
\mathrm{E}\left[(X_{t+h}-X_t)g(X_t)\right] &= \mathrm{E}_{\mathbb{Q}}\left[G^{-1}g(X_t)\delta^X(\mathbf{1}_{[t,t+h]})\right] \\
&= \mathrm{E}_{\mathbb{Q}}\left[G^{-1}\langle \mathbf{1}_{[t,t+h]}, D^X g(X_t)\rangle_{\mathcal{H}}\right] \\
&\qquad + \mathrm{E}_{\mathbb{Q}}\left[g(X_t)\langle \mathbf{1}_{[t,t+h]}, D^X G^{-1}\rangle_{\mathcal{H}}\right] \\
&= \mathrm{E}[g'(X_t)]\langle \mathbf{1}_{[t,t+h]}, \mathbf{1}_{[0,t]}\rangle_{\mathcal{H}} \\
&\qquad + \mathrm{E}\left[Gg(X_t)\langle \mathcal{K}_H^*\mathbf{1}_{[t,t+h]}, \mathcal{K}_H^* D^X G^{-1}\rangle_{\mathrm{L}^2}\right].
\end{aligned}
$$

But $\mathcal{K}_H^* D^X G^{-1} = D^Y G^{-1}$ (transfer principle). Since

$$G^{-1} = \exp\left(\int_0^T a_s dY_s - \frac{1}{2}\int_0^T a_s^2 ds\right),$$

we have

$$\begin{aligned} G \times D_t^Y(G^{-1}) &= a_t + \int_0^T D_t^Y a_s\, dY_s - \int_0^T a_s\, D_t^Y a_s\, ds \\ &= a_t + \int_0^T D_t^Y a_s\, dW_s. \end{aligned}$$

Moreover

$$\int_0^T D_s^Y a_r\, dW_r = \int_0^T (\mathcal{K}_H^* D_s^X a)(r)\, dW_r = \int_0^T D_s^X a_r \delta B_r := \Phi(s),$$

and

$$(\mathcal{K}_H^* \mathbf{1}_{[0,t]})(s) = K_H(t,s)\mathbf{1}_{[0,t]}(s).$$

Therefore

$$\begin{aligned} \langle \mathcal{K}_H^* \mathbf{1}_{[t,t+h]}, G\mathcal{K}_H^* D^X G^{-1}\rangle_{\mathrm{L}^2} &= (\mathcal{K}_H a)(t+h) - (\mathcal{K}_H a)(t) \\ &\quad + (\mathcal{K}_H \Phi)(t+h) - (\mathcal{K}_H \Phi)(t) \\ &= \int_t^{t+h} b(X_u) du + (\mathcal{K}_H \Phi)(t+h) - (\mathcal{K}_H \Phi)(t). \end{aligned}$$

By the stochastic Fubini theorem, we have $(\mathcal{O}_H \Phi)(t) = \int_0^T (\mathcal{O}_H D_\cdot^X a_r)(t) \delta B_r$. We set

$$\beta_r^H(t) = (\mathcal{O}_H D_\cdot^X a_r)(t) = \left(\mathcal{O}_H \int_0^r b'(X_s)\mathbf{1}_{s\geq \cdot}\, ds\right)(t).$$

We then deduce

$$\begin{aligned} \mathrm{E}\left[(X_{t+h} - X_t) g(X_t)\right] &= \mathrm{E}[g'(X_t)]\langle \mathbf{1}_{[t,t+h]}, \mathbf{1}_{[0,t]}\rangle_{\mathcal{H}} \\ &+ \mathrm{E}\left[g(X_t)\left(\int_t^{t+h} b(X_s) ds + \int_t^{t+h}\int_0^T \beta_r^H(s)\delta B_r ds\right)\right]. \end{aligned} \tag{24}$$

By developing $\mathrm{E}[X_t\, g(X_t)]$ as in (24), we obtain

$$t^{2H}\, \mathrm{E}[g'(X_t)] = \mathrm{E}\left[g(X_t)\left(X_t - \int_0^t b(X_s) ds - \int_0^t \int_0^T \beta_r^H(s)\delta B_r ds\right)\right].$$

Then:

$$
\begin{aligned}
&\mathrm{E}[\Delta_h X_t \,|\, X_t] \\
&= h^{-1} \langle \mathbf{1}_{[t,t+h]}, \mathbf{1}_{[0,t]} \rangle_{\mathcal{H}} \left(X_t - \mathrm{E}\left[\int_0^t b(X_s)ds + \int_0^t \int_0^T \beta_r^H(s) \delta B_r ds \,\middle|\, X_t \right] \right) \\
&\quad + h^{-1}\, \mathrm{E}\left[\int_t^{t+h} b(X_s)ds + \int_t^{t+h} \int_0^T \beta_r^H(s) \delta B_r ds \,\middle|\, X_t \right].
\end{aligned}
$$

We deduce that $\mathrm{E}[\Delta_h X_t \,|\, X_t]$ converges in probability, as $h \downarrow 0$, to

$$
\begin{aligned}
b(X_t) + \frac{H}{t} X_t - \frac{H}{t} \mathrm{E}\Bigg[\int_0^t b(X_s)ds &+ \int_0^t \int_0^T \beta_r^H(s) \delta B_r ds \\
&- \int_0^T \beta_r^H(t) \delta B_r \,\Bigg|\, X_t \Bigg].
\end{aligned}
$$

Since $\lim_{h\downarrow 0} \mathrm{E}[\Delta_h X_t \,|\, X_t]$ does not depend on T, we finally obtain (23) in the particular case where $\sigma \equiv 1$, by letting $T \downarrow t$.

Second step. Assume that σ does not vanish. Set $Y_t = h(X_t)$ where $h(x) = \int_0^x \frac{dy}{\sigma(y)}$. Using the change of variable formula, we obtain that Y verifies

$$
Y_t = y_0 + B_t + \int_0^t \frac{b}{\sigma} \circ h^{-1}(Y_s) ds, \quad t \in [0, T].
$$

Since, on the one hand, the σ-fields generated by X_t and Y_t are the same and, on the other hand, X has α-Hölder continuous paths with $\alpha > 1/2$, we have

$$
D_+^{\mathcal{T}} X_t = \sigma(X_t) D_+^{\mathcal{T}} Y_t.
$$

The expression (23) is then a consequence of the first step of the proof.

□

REMARK 23. When σ does not vanish and $b \equiv r\,\sigma$ with $r \in \mathbb{R}$, we can apply either Proposition 21 or Theorem 22 to compute $D_\pm^{\mathcal{T}} X_t$. Of course, the conclusions are the same. Indeed, since we have, in this case, $b'\sigma - b\sigma' \equiv 0$ and $\int_0^{X_t} \frac{dy}{\sigma(y)} = B_t + r\,t$ (since $X_t = f(B_t + r\,t)$ with f verifying $f' = \sigma \circ f$), formula (23) can be simplified in (22).

Compared to the case where σ and b are proportional, it is here more difficult to decide if the present (that is the collection of σ-fields generated by X_t) is discriminating or not.

In the framework of the stochastic embedding of dynamical systems introduced in [3], the set of processes, called set of Nelson differentiable processes, which satisfy the equality between a stochastic forward and stochastic backward derivatives plays a fundamental role (see [4], Chapters 3 and 7). We stress on the fact that solutions of stochastic differentiable equations driven by a fractional Brownian motion with $H > 1/2$ provide examples of non absolutely continuous Nelson differentiable processes.

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